Indian Statistical Institute, Bangalore Centre

Solution set of M.Math II Year, End-Sem Examination 2012

Fourier Analysis

1. Let H be the real Hilbert transform given by

$$(Hf)(x) = CPV \int_{-\infty}^{\infty} \frac{f(y)}{x-y}, dy.$$

Calculate $H_{\chi_{[a,b]}}$ for $-\infty < a < b < \infty$.

Proof.

$$\begin{aligned} H_{\chi_{[a,b]}}(x) &= CPV \int_{-\infty}^{\infty} \frac{\chi_{[a,b]}(y)}{x-y}\ dy \\ &= CPV \int_a^b \frac{1}{x-y}\ dy \\ &= \lim_{\epsilon \to 0^+} \int_a^{x-\epsilon} \frac{1}{x-y}\ dy + \int_{x+\epsilon}^b \frac{1}{x-y}\ dy \\ &= \lim_{\epsilon \to 0^+} \left[[\log|x-y|]_a^{x-\epsilon} + [\log|x-y|]_{x+\epsilon}^b \right] \\ &= \log\left|\frac{x-b}{x-a}\right|. \end{aligned}$$

□

2. Let Y be a closed linear subspace of $L^1(\mathbb{R}^n)$. Assume that $Y * L^1(\mathbb{R}^n) \subset Y$. If $f \in Y$, then show that $\tau_s f \in Y$ where $(\tau_s f)(x) = f(x-s)$.

Proof. Let u run through an approximate identity. If $f \in Y$, then $u_s * f \in Y$. But $u_s * f = (u * f)_s \to \tau_s f$, since $u * f \to f$, and therefore $\tau_s f \in Y$. □

3. Let $f_k(x) = x^k e^{-\frac{x^2}{2}}$ for $k = 0, 1, 2, \ldots.$ and $x \in \mathbb{R}$. Find a relation between $\hat{f}_{k+1}(s)$ and $s\hat{f}_k(s)$.

Proof. Note that $f_{k+1}(x) = x f_k(x)$. Recall that $\widehat{e^{-\frac{x^2}{2}}}(s) = \sqrt{2\pi} e^{-\frac{s^2}{2}}$ and $\widehat{x^n f(x)}(s) = i^n \frac{d^n}{ds^n} \hat{f}(s)$. Using these facts, we have

$$\hat{f}_{k+1}(s) = i^{k+1} \frac{d^{k+1}}{ds^{k+1}} \widehat{e^{-\frac{x^2}{2}}}(s) = \sqrt{2\pi}\ i^{k+1} \frac{d^{k+1}}{ds^{k+1}} e^{-\frac{s^2}{2}}$$

$$= \sqrt{2\pi}\ i^{k+1}\frac{d^k}{ds^k}e^{-\frac{s^2}{2}}(-s)$$
$$= -\sqrt{2\pi}\ is\big[i^k\frac{d^k}{ds^k}e^{-\frac{s^2}{2}}\big]$$
$$= -\sqrt{2\pi}\ is\hat{f}_k(s).$$

□

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be in $L^1_{\text{loc}}(\mathbb{R})$ with compact support. If the maximal function Mf is in $L^1(\mathbb{R})$, then show that $f = 0$.

   (b) Let $g \in L^1(\mathbb{R})$. If $Mg \in L^1(\mathbb{R})$, then show that $g = 0$. Note that g is not assumed to have compact support.

Proof. Let $0 \neq f \in L^1_{\text{loc}}(\mathbb{R})$ with $\text{supp}(f) \subset B_R$ for some $R > 0$ and $\|f\|_L^1 > 0$. Note that for $|x| > R$,

$$Mf(x) \geq \frac{1}{|B_{|x|}|}\int_{B_{|x|}} |f(y)|\ dy = \frac{C\|f\|_{L^1}}{(|x| + R)} \notin L^1(\mathbb{R}).$$

In general, we can approximate f by $f1_{|x|>R}$ and there exists R such that

$$\|f - f1_{|x|>R}\|_L^1 \leq \epsilon.$$

Then

$$Mf(x) \geq \frac{1}{|B_{|x|}|}\int_{B_{|x|}} |f(y)|\ dy = \frac{C\|f1_{|x|>R}\|_{L^1} - \epsilon}{(|x| + R)} \notin L^1(\mathbb{R}).$$

□

5. Let $\psi \in L^2(\mathbb{R})$ satisfy $\int dw|\frac{|\ \hat{\psi}(w)|^2}{|w|} < \infty$. Define $\psi_{(a,b)}$ for $a > 0$ and $b \in \mathbb{R}$ by $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi(\frac{t-b}{a})$. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, define $(Wf)(a, b) = \langle f, \psi_{a,b}\rangle$. Show that

$$\int_0^\infty \frac{da}{a^2}\int_{\mathbb{R}} db\ |(Wf)(a, b)|^2 = C_\psi\|f\|^2_{L^2}.$$

Proof.

$$\int_0^\infty \frac{da}{a^2}\int_{\mathbb{R}} db\ |(Wf)(a, b)|^2 = \int_0^\infty \frac{da}{a^2}\int_{\mathbb{R}} db\ \langle f, \psi_{(a,b)}\rangle\langle\psi_{(a,b)}, f\rangle \qquad (1)$$

$$= \int_0^\infty \frac{da}{a^2} \int_{\mathbb{R}} db \, \Big[\int dx \hat{f}(x) |a|^{\frac{1}{2}} e^{-ibx} \overline{\hat{\psi}(ax)} \Big] \Big[\int dy \overline{\hat{f}(y)} |a|^{\frac{1}{2}} e^{iby} \hat{\psi}(ay) \Big]. \qquad (2)$$

The expression between the first pair of brackets can be viewed as $(2\pi)^{\frac{1}{2}}$ times the Fourier transform of $F_a(x) = |a|^{\frac{1}{2}} \hat{f}(x) \overline{\hat{\psi}(ax)}$; the second has a similar interpretation as $(2\pi)^{\frac{1}{2}}$ times the complex conjugate of $F_a(x)$. From equation (1) and by the unitarity of the Fourier transform it follows that

$$\begin{aligned}
&\int_0^\infty \frac{da}{a^2} \int_{\mathbb{R}} db \, |(Wf)(a,b)|^2 = 2\pi \int \frac{da}{a^2} \int dx F_a(x) \overline{F_a(x)} \\
&= 2\pi \int \frac{da}{a} \int dx \hat{f}(x) \overline{\hat{f}(x)} |\hat{\psi}(ax)|^2 \\
&= 2\pi \int dx \hat{f}(x) \overline{\hat{f}(x)} \int \frac{da}{a} |\hat{\psi}(ax)|^2 \quad \text{(By Fubini's theorem)} \\
&= C_\psi \|f\|_{L^2}^2.
\end{aligned}$$

$\square$